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JULIA SET AND SOME OF ITS PROPERTIES

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ABSTRACT

In this paper we have studied several properties of the Julia sets for the complex polynomial of the form $z^2 + c$. We have also discussed the stability nature of its fixed points and periodic points in some situations depending on the value of the parameter c.

KEYWORDS: Fixed Point, Periodic Point, Iteration of a Map, Stability of Fixed and Periodic Points

1. INTRODUCTION

Julia sets are amongst the most frequently pictured fractals, combining both aesthetic and actual beauty. The modern day interest in Julia sets and related mathematics began in the year 1920, which was initiated by French mathematician Gaston Julia. In 1918 he wrote a paper titled "*M´emoire sur l'iteration des functions rationelles*" (A Note on the Iteration of Rational Functions) [11] where he first introduced the modern idea of a Julia set. In this paper Julia gave a precise description of the set of those points of the complex plane whose orbits under the iteration of a rational function stayed bounded. Interest in the subject flourished over the next 10 years and many other well-known mathematicians like Harald Cramer began to study Julia set. Despite the lack of computing machine available at that time he was able to become the first man to approximate an image of a Julia set. Due to the lack of computing machine the progress of research in this line slowed down and after some years Julia's work was forgotten by the mathematical community. It was the France mathematician Benoit Mandelbrot who brought back Julia's work around 1977. With the aid of computer graphics he showed that Julia's work is a source of some of the most beautiful fractals known today.

A very brief historical note will be useful to explain the role of iteration of rational maps on the complex plane which is the central theme of this paper. The study of iteration of maps begins with the Newton's method. In 1870's Cayley and Schröder independently studied Newton's method over the complex plane [7]. Julia's interest on iteration of maps apparently was motivated by the 1879 paper "*The Newton-Fourier Imaginary Problem*" by Sir Arthur Cayley [14]. Cayley was studying the equation $f(z) = z^3 + c = 0$, using the Newton- Fourier iterative method to find the roots. In this iterative method one can approach to a root as a limiting value for a certain starting value of z. Since there are three roots, Cayley tried to find out a method by which it can be ascertained to which of these three roots the iteration will converge for a given starting value of z. But he could not solve this problem. Over the following decades other mathematicians continued to build on this work, but their work failed to create a general and global theory that could describe the behavior of all rational functions on the entire complex plane.

In 1915 the French Academy of Sciences announced that it would award the 1918 *Grand Prix des Sciences Mathematique* for such a general and global theory. The two mathematicians who succeeded in creating the modern theory of complex dynamics were Gaston Julia and Pierre Fatou. Julia won the contest and Fatou, despite withdrawing his work from the contest, received a second prize. Both saw that iteration of any function partitioned the extended complex plane into a region where the iterates are equicontinuous and a region where they are not. Today, for any non-constant rational function f, the maximal open subsets of the extended complex plane on which the family of iterates of the function f, i.e., $\{f^n : n \geq 1\}$ is equicontinuous is called Fatou set of f, denoted by F or $F(f)$, while the complement of F in the extended complex plane is called the Julia set of f, denoted by J or $J(f)$. This concept is the primary basis of this paper.

With the arrival of modern sophisticated computers in 1980s the above mentioned intricate problem of course could be handled and the research in this area got momentum. Let us see little bit in detail the mathematical theory behind the above mentioned problem.

Consider $F(z) = z - \dfrac{f(z)}{f'(z)}$.

Suppose z_0 be a root of $f(z)$, i.e. $f(z_0) = 0$. Then we have $F(z_0) = z_0$, which shows that z_0 is a fixed point of $F(z)$.

Thus, any complex number z is a root of $f(z) = 0$ if and only if it is a fixed point of $F(z)$.

Now, if we consider $F(z) = z - \dfrac{z^3 - 1}{3z^2} = \dfrac{2z^3 - 1}{3z^2}$.

Then clearly the roots of $f(z)$ are the cube roots of unity, viz., $z_1 = 1$, $z_2 = e^{\frac{2\pi i}{3}}$ and $z_3 = e^{\frac{4\pi i}{3}}$. These are also attracting fixed point of $F(z)$. Therefore, the problem of interest is now converted to find out the basin of attraction of these fixed points of $F(z)$. Using a computer to graph this situation in the complex plane we get the image as shown in the figure 1.1. Here, the basin of attractions for z_1, z_2 and z_3 are colored respectively by blue, green and yellow. Note that the basin boundary in this figure is nothing but the Julia set for the function $F(z)$.

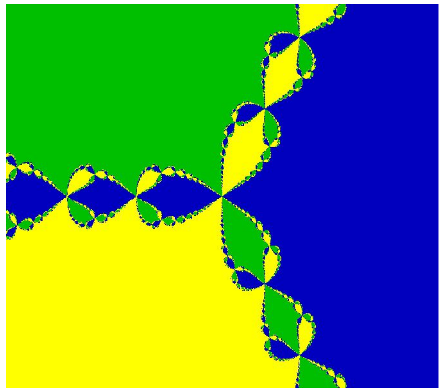


Figure 1.1

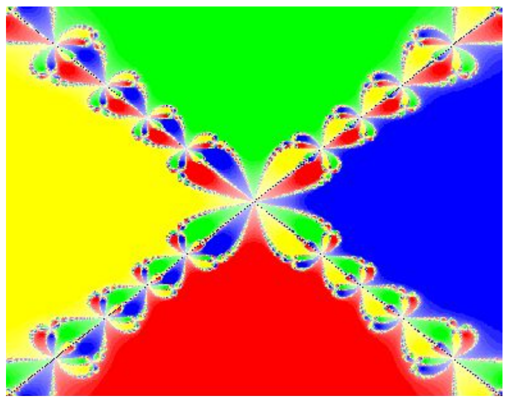


Figure 1.2

With a similar fashion one can find the basin of attraction for the roots of the equation $z^4 - 1 = 0$ as shown in the figure 1.2, where the region colored in blue, yellow, green and red are respective basin of attraction for the roots of $f(z)$ or fixed points of $F(z)$ which are 1, -1, i and $-i$.

For fundamental work relevant to study of iteration of rational polynomials one may refer Beard on [1], Devaney [5, 6] and Steinmetz [18]. For computing Hausedorff dimension of the Julia sets one can see [2]. Julia set is a classical example of fractal, which has lots of application in medical science, study of weather system, biology, wireless communications etc. Interested readers can go through [10, 12] for finding such applications. Julia set which is a fractal is a great tool for animating the various aspect of world around us, it is used to animate feature films like '*Star Track-II*' [12]. Moreover, for applications of Julia sets in wavelet analysis and in the study the large scale distribution of galaxies in the observed universe one can see [13] and [16] respectively.

The rest of the paper is organized as follows. In section-2, we provide a review of preliminary concepts and definitions along with some examples to clarify those concepts and definitions. Section -3 and 4 deals with some dynamical and structural properties related to Julia set. In section 5 we have described about how to visualize the Julia set. Finally, we have given our conclusions in section 6.

2. SOME PRELIMINARIES

As discussions on Julia set involve the extended complex plane (that is infinity is included), we have given a brief review of how it can be achieved.

Stereographic Projection: Stereographic projection is a method to create a planar map of a sphere. In the figure given below, we draw a sphere of radius one whose south pole touches the complex plane at the origin. To each point of the sphere there is a corresponding point in the complex plane. To be specific, the ray from the North Pole through the point (x, y, z) on the sphere will intersect the complex plane at a point denoted by $u + iv$ enabling us to define a map as

$$P(x, y, z) = \frac{2}{2-z}(x+iy)$$

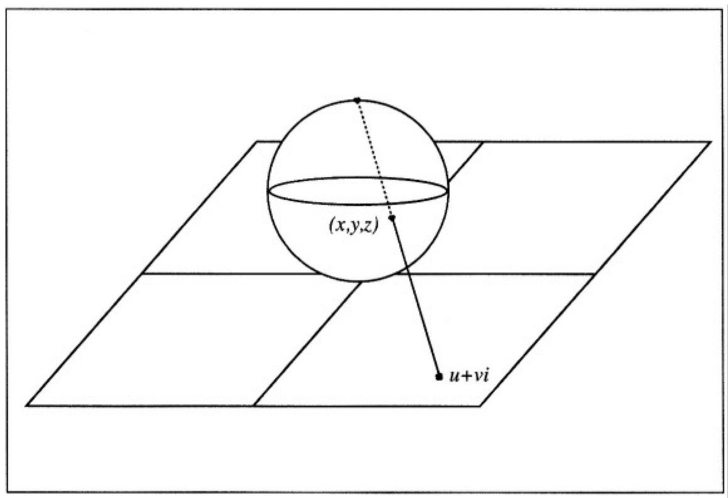


Figure 2.1: Stereographic Projection

Hence, $u = \frac{2}{2-z}x$ and $v = \frac{2}{2-z}y$.

The inverse of this transformation is given by

$$P^{-1}(u+iv) = \frac{1}{u^2+v^2+4}\left(4v, 4u, 2u^2+2v^2\right).$$

Which gives the point (x, y, z) on the sphere corresponding to the point $u+iv$ of the complex plane C The z-component of this inverse transformation is

$$z = \frac{2(u^2+v^2)}{u^2+v^2+4}$$

When $u + iv$ tends to infinity, i.e., u^2+v^2 goes without bounds the z-component tends to 2 since, $\lim_{u^2+v^2\to\infty}\frac{2(u^2+v^2)}{u^2+v^2+4} = 2$ But, the only point on the sphere whose z-component is 2 is the North Pole i.e., the point $(0,0,2)$. Hence, the North Pole can be regarded as the point at infinity.

In this way the extended complex plane can be considered as a sphere, called Riemannian sphere. We denote it by $\hat{C}$, i.e., $\hat{C} = C \cup \{\infty\}$.

Let σ be the metric on the Riemannian sphere which simply denote the length of the chord between any two points on the sphere. This metric is called spherical metric or the chordal metric, and for any two points $z, w \in \hat{C}$, it is defined as

$$\sigma(z,w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$$

When $w \to \infty$, then $\sigma(z,w) = \dfrac{2}{\sqrt{1+|z|^2}}$

With this metric $\hat{C}$ is a compact metric space [14].

Consider an arbitrary polynomial $f : \hat{C} \to \hat{C}$. Let f^n denote the n[th] iterate of f , that is , f composed with itself n times. For each point $z_0 \in \hat{C}$, we are interested in the behavior of the sequence $z_0, f(z_0), f^2(z_0), \cdots, f^n(z_0), \cdots$, and in particular, what happens as n goes to infinity.

Now we review several definitions related to the iterations of rational functions which are relevant to carry our study.

Definition 2.1: A function f is called analytic in $\hat{C}$ if its derivative exists at each point in $\hat{C}$**.**

Definition 2.2: A point $z_0 \in \hat{C}$ is called critical point for the function f if $f'(z_0)=0$.

Definition 2.3: A point $z_0 \in \hat{C}$ is called periodic point of f if $f^n(z_0) = z_0$ for some integer $n \geq 1$. The smallest n with this property is called the period of z_0. Thus, the periodic points of z_0 are the zeros of the function $F(z_0, f) = f^n(z_0) - z_0$.

A periodic point with period one is termed as fixed point of f i.e., z_0 is a fixed point of f it $f(z_0)=z_0$.

Definition 2.4: The multiplier (or Eigen value, derivative) λ of a rational map f iterated n times, at the periodic point z_0 is defined as:

$$\lambda = \begin{cases} {f^n}'(z_0), & if\ z_0 \neq \infty \\ \dfrac{1}{{f^n}'(z_0)}, & if\ z_0 = \infty \end{cases}$$

Where $f^{n'}(z_0)$ is the first derivative of f^n with respect to z at z_0.

Note that, the multiplier is same at all periodic points of a given orbit. Therefore it can be regarded as multiplier of the periodic orbit.

The absolute value of the multiplier is called the stability index of the periodic point. It is used to check the stability of periodic points.

Definition 2.5: A periodic point z_0 is called attracting periodic point if $|\lambda| < 1$, and is repelling if $|\lambda| > 1$. It is called indifferent when $|\lambda| = 1$.

Example 2.1: Here we discuss the stability of periodic points of the function $f(z) = z^3$. The fixed points are zeros of

$$f(z) - z = 0$$

Which are found to be $z_1 = 0,\ z_2 = 1,\ z_3 = -1$. Now, $f'(z) = 3z^2$, so $|f'(z_1)| = 0 < 1$, therefore, $z_1 = 0$ is a stable fixed point. Likewise, $z_2 = 1$ and $z_3 = -1$ are both unstable fixed points.

To find the 2-periodic points we need to solve the equation;

$$f^2(z) - z = 0$$

The solutions to this equation are $z_1 = 0$ and the eighth roots of unity which are $1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7$ where, $\omega = \cos\left(\frac{2\pi}{8}\right) + i\sin\left(\frac{2\pi}{8}\right) = \frac{1}{\sqrt{2}}(1+i)$.

Since we have,

(i) $f(\omega) = \omega^3,\ \ f^2(\omega) = f(\omega^3) = \omega^9 = \omega$

(ii) $f(\omega^2) = \omega^6,\ \ f^2(\omega^2) = f(\omega^6) = \omega^{18} = \omega^2$

(iii) $f(\omega^5) = \omega^{15} = \omega^7,\ \ f^2(\omega^5) = f(\omega^7) = \omega^{21} = \omega^5$

So, the 2-cycles are $\{\omega, \omega^3\}$, $\{\omega^2, \omega^6\}$ and $\{\omega^5, \omega^7\}$.

As $z_1 = 0,\ z_2 = 1,\ z_3 = \omega^4 = -1$ are fixed points of f and we have already discussed their stability we consider the points $\{\omega^k : k = 1, 2, 3, 5, 6, 7\}$.

Now, $\left| f'(\omega^k) f'(f(\omega^k)) \right| = \left|(9\omega^{8k})\right| = 9 > 1$. So each eighth root of unity is unstable.

After clarifying the definitions with examples now we go back to the study of the maps of the form

$$f_c(z) = z^2 + c \tag{1}$$

For different values of the parameter $c \in \hat{C}$ the reason for choosing the above map is that every quadratic polynomial is linear conjugate to a map of this form and so it is representative of all quadratic polynomials. An important property of the Julia set of a rational function is that it is equal to the closure of its repelling periodic points. A rational function is defined as $R(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ is polynomials with roots distinct from each other. Every polynomial of the form $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ can be regarded as a rational function. Therefore, the conceptually easiest way to define Julia set is as follows:

Definition 2.6: Let c be any complex number. The smallest closed set in the complex plane that contains all repelling fixed points and all repelling periodic points of the map $f_c(z) = z^2 + c$ is called Julia set of the map f_c, and it is denoted by J_c.

For polynomials of dgree at least two, ∞ is always an attacting fixed point in the light of our earlier discussion of stereographic projection. Some authors such as Goodson [9] use this fact to define the Julia set of polynomials as:

Definition 2.7: The basin of attraction of ∞ for the polynomial $f(z)$ having degree at least two, is the open set

$$B_f(\infty) = \{z \in \hat{C} : f^n(z) \to \infty \ as\ n \to \infty\}.$$

Definition 2.8: The Julia set $J(f)$ of the polynomial $f(z)$ having dgree at least 2, is the boundary of the open set $B_f(\infty)$, i.e., the set $\overline{B}_f(\infty) - B_f(\infty)$.

Definition 2.9 : The set $K(f)$ of all those points of $\hat{C}$ which do not converge to ∞ under iteration of the polynomial $f(z)$ having degree at least two, is called the filled in Juia set of $f(z)$, i.e.,

$$K(f) = C - B_f(\infty).$$

Note that, the Julia set $J(f)$ is also the boundary $\partial K(f)$ of the set $K(f)$. Thus, for each $z_0 \in J(f)$ there is an open sphere $S_r(z_0)$ with centre at z_0 and radius $r > 0$, there is a point $u \in S_r(z_0)$ such that iterates of u under f converge to infinity as well as another point $v \in S_r(z_0)$ such that iterates of v under f do not converge to infinity.

Definition 2.10: The complement of the Julia set $J(f)$ of the polynomial $f(z)$ is called the Fatou set. It is denoted by $F(f)$, i.e., $F(f) = C - J(f)$.

Theorem 2.1: For any polynomial, $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$; $a_n \neq 0$, there is a real number r such that if $|z| > r$, then $|f(z)| \geq 2|z|$. Furthermore, the iterates of f are either bounded or tends to infinity.

Proof: Since $a_n \neq 0$ we choose $r = \sqrt[(n-1)]{\frac{4}{|a_n|}}$. Thus, for $|z| > r$ we have

$$|z| \geq \sqrt[(n-1)]{\frac{4}{|a_n|}}$$

$$\Rightarrow |z|^{n-1} \geq \frac{4}{|a_n|} \Rightarrow \frac{1}{2}|a_n||z|^{n-1} \geq 2 \Rightarrow \frac{1}{2}|a_n||z|^{n} \geq 2|z|$$

Now we can make r sufficiently large to ensure that when $|z| \geq r$,

$$|a_{n-1}||z|^{n-1} + \cdots + |a_1||z| + |a_0| \leq \frac{1}{2}|a_n||z|^n$$

Thus

$$|f(z)| = |a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0|$$

$$\geq |a_n||z|^n - \left(|a_{n-1}||z|^n + |a_{n-1}||z|^{n-1} + \cdots + |a_1||z| + |a_0|\right)$$

$$\geq |a_n||z|^n - \frac{1}{2}|a_n||z|^n \geq \frac{1}{2}|a_n||z|^n \geq 2|z|$$

Now, let us consider the set $\{f^n(z) : n \in N\}$ of all iterates of f.

If $|f^n(z)| \leq r$ for all $n \in N$ then clearly the iterates of f are bounded.

Otherwise, if $|f^m(z)| \geq r$ for some $m \in N$ then we have

$$|f^{m+k}(z)| = |f(f^{m+k-1}(z))|$$

$$\geq 2|f^{m+k-1}(z)|$$

$$\geq 2^2|f^{m+k-2}(z)|$$

$$\vdots$$

$$\geq 2^k|f^m(z)|$$

Thus, $\left|f^{m+k}(z)\right| \geq 2^k \left|f^m(z)\right| \geq 2^k r$, so $\left|f^k(z)\right| \to \infty$ as $k \to \infty$, i.e. the iterates of f tends to infinity.

3. SOME DYNAMICAL PROPERTIES RELATED TO THE JUIA SET

In this section we discuss some dynamical properties of the Julia sets J_c for different values of the parameter c.

Let us start with $f_c(z) = z^2 + c$ where c is an arbitrary fixed complex parameter.

First we consider the case $c = 0$ i.e., $f_0(z) = z^2$.

The fixed points of f_0 are given by

$$f_0(z) = z \;\; i.e. \quad z = 0, 1$$

As, $f_0'(z) = 2z$, $z = 0$ is attracting and $z = 1$ is repelling fixed points of f_0.

Let $z_0 = re^{i\theta}$, then the orbit of z_0 under f_0 is given by:

$$z_0 = re^{i\theta}$$

$$z_1 = r^2 e^{i(2\theta)}$$

$$\vdots$$

$$z_n = r^{2^n} e^{i(2^n\theta)}$$

Thus, the image of z_0 under f_0 is obtained by squaring the magnitude of z_0 and doubling the angle (argument) of z_0 . Therefore, the behavior of the orbit of z_0 depends on its magnitude.

If $r < 1$, then $r^{2^n} \to 0$ as $n \to \infty$, therefore, the orbit of z_0 will tend to the origin which is an attracting fixed point of f_0.

If $r = 1$, that is, if z_0 is on the unit circle $|z| = 1$, then $r^{2^n} = 1$, so the orbit of z_0 will remain on the unit circle and on each iteration f_0 will doubles the angle (argument) of z_0.

If $r > 1$, then $r^{2^n} \to \infty$ as $n \to \infty$, and so the orbit of z_0 will tends to ∞, so the orbit in this case will be unbounded. This situation has been shown in the following cob-web diagram.

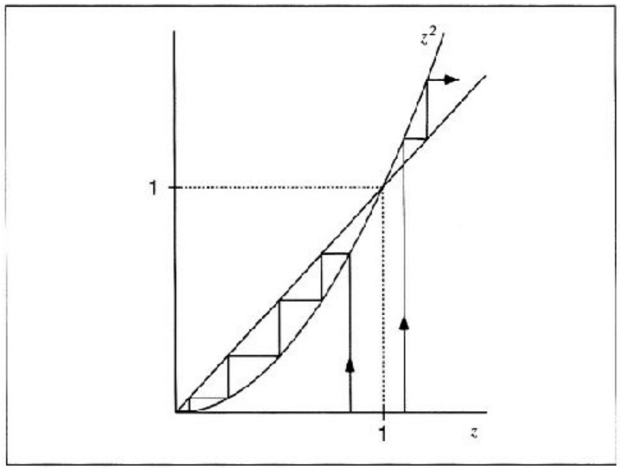


Figure 3.1: The Cob-Web Diagram for f_0

Further, the following Table 1 gives the iterated values of z_0 under f_0 in each of the above mentioned situations.

Table 3.1

	Z_0	$Z_1 = f_0(z_0)$	$Z_2 = f_0(z_1)$	$Z_3 = f_0(z_2)$	$Z_4 = f_0(z_3)$	$Z_5 = f_0(z_4)$
$\lvert z \rvert < 1$	Modulus = 0.8 Argument = 10°	Modulus = 0.64 Argument = 20°	Modulus = 0.4096 Argument = 40°	Modulus = 0.1678 Argument = 80°	Modulus = 0.0281 Argument = 160°	Modulus = 0.0008 Argument = 320°
$\lvert z \rvert = 1$	Modulus = 1 Argument = 10°	Modulus = 1 Argument = 20°	Modulus = 1 Argument = 40°	Modulus = 1 Argument = 80°	Modulus = 1 Argument = 160°	Modulus = 1 Argument = 320°
$\lvert z \rvert > 1$	Modulus r = 1.5 Argument Θ = 50°	Modulus = 2.25 Argument = 100°	Modulus = 5.06 Argument = 200°	Modulus = 25.6 Argument = 40°	Modulus = 655.36 Argument = 80°	Modulus =431440 Argument = 160°

The iteration of three initial points mentioned in the Table-1 is graphically represented in figure-4.

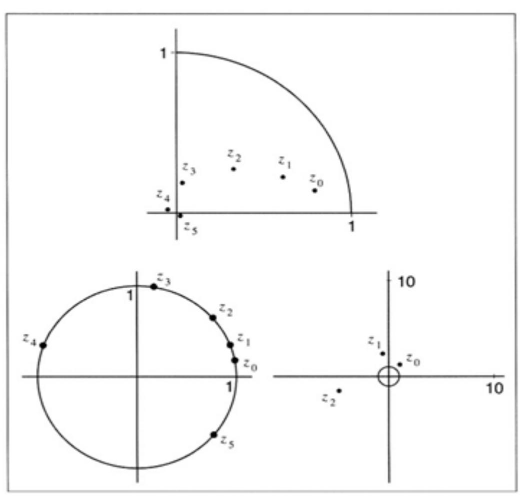


Figure 3.2: Iteration of the Point z_0 under f_0 as Shown in Table-3.1

Since the doubling map on the unit circle is chaotic [4], f_0 is chaotic on the unit circle. This circle is the boundary between the set of initial points in the complex plane with itineraries that approaches the origin and those with itineraries that go to infinity. This separating curve (circle) is the Julia set of f_0 and clearly the interesting dynamics of f_0 takes place on it. At this point it is noteworthy that although f_0 exhibits chaotic behaviour on its Julia set, its study may lead one to believe that Julia sets are nice smooth curves. In fact, a variation in the parameter c give rise to quadratic maps whose Julia sets are fractals. Ruelle [17] has shown that for f_c with small $|c|$, the fractal dimension of its corresponding Julia set is approximately

$$d_c = 1 + \frac{|c|^2}{4\log 2}$$

And so is indeed a fractal in these cases.

Next, it is to be noted that the fixed points of f_c are given by

$$z = \frac{1 \pm \sqrt{1-4c}}{2}$$

Clearly, number of fixed points will differ depending on $c = \frac{1}{4}$ or $c \neq \frac{1}{4}$

Now, we discuss the nature of attracting and repelling fixed points of (1) through the following theorems:

Theorem 3.1: $f_{\frac{1}{4}}$ Have neither attracting nor repelling fixed points

Proof: Fixed points of $f_{\frac{1}{4}}$ is given by the quadratic equation:

$$f_{\frac{1}{4}}(z) = z \qquad \Rightarrow z^2 - z + \frac{1}{4} = 0 \Rightarrow z = \frac{1}{2}, \frac{1}{2}$$

Thus, the only fixed point of $f_{\frac{1}{4}}$ is at $z = \frac{1}{2}$.

Now, $f_{\frac{1}{4}}'\left(\frac{1}{2}\right) = 1$, which shows that $z = \frac{1}{2}$ is indifferent fixed point of $f_{\frac{1}{4}}$.

Therefore, $f_{\frac{1}{4}}$ has neither attracting nor repelling fixed point. ■

Theorem 3.2: If $c \neq \frac{1}{4}$, then at least one fixed point of f_c is repelling.

Proof: Fixed points of f_c is given by the quadratic equation:

$f_c(z) = z \quad \Rightarrow z^2 + c = z$

Thus, the fixed points of (1) are the roots z_1 and z_2 of the quadratic equation

$z^2 - z + c = 0$ And therefore, $z_1 = \frac{1}{2} + \frac{1}{2}\sqrt{1-4c}$ and $z_2 = \frac{1}{2} - \frac{1}{2}\sqrt{1-4c}$.

From this it is clear that for $c \neq \frac{1}{4}$, f_c has two distinct fixed points viz., z_1 and z_2 .

Now $f_c'(z) = 2z$, therefore,

$\lambda_1 = f_c'(z_1) = 1 + \sqrt{1-4c} = 1+\mu$, and $\lambda_2 = f_c'(z_2) = 1 - \sqrt{1-4c} = 1 - \mu$,

Where $\mu = \sqrt{1 - 4c}$

As $c \neq \frac{1}{4}$, $\mu \neq 0$.

Consider the unit circle

$|\ z\ | = 2 \ \cdots$ (2)

Two cases might arise in case of position of a point with respect to the circle (2) depending on the value of μ which are discussed below:

Case 1: When μ lies outside the circle (2).

In this case both the distances of μ and $-\mu$ from 1 are greater than unity, i.e.

$|1 - \mu| > 1$ And $|1 + \mu| > 1$

$\Rightarrow |\ \lambda_2\ | > 1$ And $|\ \lambda_1\ | > 1$

Therefore, both z_1 and z_2 are repelling fixed point of (1)

Case 2: When μ lies inside or on the circle (2).

In this case, as $\mu \neq 0$ at least one of the distances of μ or $-\mu$ from 1 is greater than the unity, i.e. at least one of $|\ 1 - \mu\ |$ or $|\ 1 + \mu|$ is greater than one. Therefore, at least one of z_1 or z_2 is repelling fixed point of (1). ■

Theorem 3.3: $f_{\frac{1}{4}}(z) = z^2 + \frac{1}{4}$ has repelling 2-cycles.

Proof: It is already shown in theorem 3.1 that the only fixed point of $f_{\frac{1}{4}}(z)$ is $z=\frac{1}{2}$ which is neither attracting nor repelling.

Now the 2-cycles of $f_{\frac{1}{4}}$ are given by

$$f_{\frac{1}{4}}^{\,2}(z)=z\Rightarrow\left(z^2+\frac{1}{4}\right)^2+\frac{1}{4}=z\Rightarrow z^4+\frac{1}{2}z^2-z+\frac{5}{16}=0 \quad (3)$$

Since $\frac{1}{2}$ is fixed point of $f_{\frac{1}{4}}(z)$, it is also a fixed point of $f_{\frac{1}{4}}^{\,2}(z)$ and therefore, $\left(z-\frac{1}{2}\right)$ is a factor of left hand side of (3). On factorization we get from (3);

$$\left(z-\frac{1}{2}\right)\left(z^3+\frac{1}{2}z^2+\frac{3}{4}z-\frac{5}{8}\right)=0$$

It turns out that $\left(z-\frac{1}{2}\right)$ are again a factor of $z^3+\frac{1}{2}z^2+\frac{3}{4}z-\frac{5}{8}$, i.e. $\frac{1}{2}$ is a double root of the equation (3), which leads to the following factorization:

$$\left(z-\frac{1}{2}\right)^2\left(z^2+z+\frac{5}{4}\right)=0$$

By solving, $z=\frac{1}{2}$, $-\frac{1}{2}\pm i$ Therefore, the fixed points of $f_{\frac{1}{4}}^{\,2}(z)$ are $\frac{1}{2}, -\frac{1}{2}+i$, and $-\frac{1}{2}-i$. But, neither $-\frac{1}{2}+i$ nor $-\frac{1}{2}-i$ is a fixed point of $f_{\frac{1}{4}}(z)$, this leads us to conclude that $\left\{-\frac{1}{2}+i\ ,\ -\frac{1}{2}-i\right\}$ is a 2-cycle of $f_{\frac{1}{4}}(z)$.

To show this 2-cycle is repelling we first find the multiplier λ and then use the definition 1.5 as follows:

$$|\ \lambda\ |=\left|\left(f_{\frac{1}{4}}^{\,2}\right)'\left(-\frac{1}{2}+i\right)\right|=\left|f_{\frac{1}{4}}'\left(f_{\frac{1}{4}}\left(-\frac{1}{2}+i\right)\right)f_{\frac{1}{4}}'\left(-\frac{1}{2}+i\right)\right|$$

$$=\left|f_{\frac{1}{4}}'\left(-\frac{1}{2}-i\right)f_{\frac{1}{4}}'\left(-\frac{1}{2}+i\right)\right|\ ;\ \text{Since, }\ f_{\frac{1}{4}}\left(-\frac{1}{2}+i\right)=-\frac{1}{2}-i$$

$$=\left|2\left(-\frac{1}{2}-i\right).2\left(-\frac{1}{2}+i\right)\right|\ ;\ \text{As, }\ f_{\frac{1}{4}}'(z)=2z$$

$$= \left| (-1-2i)(-1+2i) \right| = 5$$

As $|\lambda| > 1$, the 2-cycle $\left\{ -\frac{1}{2}+i \, , -\frac{1}{2}-i \right\}$ is repelling. ■

4. SOME PROPERTIES RELATED TO THE STRUCTURE OF JULIA SETS

In this section we will prove some results relating to the structure of Julia sets.

Property 4.1: For every complex number c, the Julia set of f_c is non-empty.

Proof: By theorem 3.2, for $c \neq \frac{1}{4}$, f_c has a repelling fixed point and by theorem 3.3, for $c = \frac{1}{4}, f_c$ has a repelling period-2 cycle. As, $J(f_c)$ i.e. Julia set of f_c is the closer of the set of all repelling periodic points of f_c hence $J(f_c) \neq \phi$. ■

Property 4.2: If $|z| > |c| + 1$, then the orbit of z for f_c is unbounded.

Proof: When $c = 0$, the iterates of z are just positive powers of z as $f_0(z) = z^2$, which is clearly unbounded for $|z| > 1$.

Suppose, $c \neq 0$ and $|z| > |c| + 1$, then

$$|f_c(z)| = |z^2 + c| = |z| \left| z + \frac{c}{z} \right| \geq |z| \left| |z| - \frac{|c|}{|z|} \right|$$

$$> |z| \left| |c| + 1 - \frac{|c|}{|c|+1} \right| = |z| \left| 1 + \left(|c| - \frac{|c|}{|c|+1} \right) \right| = r|z|$$

Where, $r = \left| 1 + \left(|c| - \frac{|c|}{|c|+1} \right) \right| > 1$ as $|c| > \frac{|c|}{|c|+1}$

It follows that $\left| f_c^n(z) \right| > r^n |z| \to \infty$ as n increases without bound. Therefore if $|z| > |c| + 1$, then the iterates of z form an increasing, unbounded sequence, so the orbit of z is unbounded. ■

Definition 4.1: A set S is called invariant under the function f if $f(S) \subseteq S$ i.e. the image of S under f is contained in S.

Property 4.3: The Julia set J of any complex polynomial f is completely invariant under f. That is, $J = f[J] = f^{-1}[J]$.

Proof: Suppose,

$z \in J \Rightarrow f^{k+1}(z) = f^{k}(f(z))$ Not tends to infinity as $k \to \infty$

Recall that J is the boundary of the basin of attraction of infinity and z is a repelling point of f.

As f is continuous at z, we can find $u_n \to z$ such that

$f^{k+1}(u_n) = f^{k}(f(u_n))$ Tends to infinity as $k \to \infty$ for all n

Therefore, $f(z)$ is not on the basin of attraction of infinity and we can find points $f(u_n)$ as close to $f(z)$ as we wish such that $f(u_n)$ are on the basin of attraction of infinity. From which one can conclude that

$$f(z) \in J \tag{4.3.1}$$

This implies that

$$f[J] \subseteq J \tag{4.3.2}$$

Next consider the pre-image z_0 of z i.e. $f(z_0) = z$. By mapping properties of f, we may find $v_n \to z_0$ with $f(v_n) = u_n$.

$\therefore f^{k}(z_0) = f^{k-1}(z)$ Not tends to infinity as $k \to \infty$

And $f^{k}(v_n) = f^{k-1}(u_n)$ tends to infinity as $k \to \infty$ for all n

Hence $z_0 \in J \Rightarrow z = f(z_0) \in f[J]$ (4.3.3)

$$\therefore J \subseteq f[J] \tag{4.3.4}$$

(4.3.2) and (4.3.4) implies that

$$J = f[J] \tag{4.3.5}$$

From (4.3.1)

$z \in f^{-1}[J]$ And therefore, $J \subseteq f^{-1}[J]$ (4.3.6)

From (4.3.3)

$f^{-1}(z) \in J$ And therefore, $f^{-1}[J] \subseteq J$ (4.3.7)

(4.3.6) and (4.3.7) implies that

$$J = f^{-1}[J] \tag{4.3.8}$$

(4.3.5) and (4.3.8) together constitute the result. ■

Property 4.4: The orbit of $z \in J(f_c)$ is bounded.

Proof: Let $z \in J(f_c)$. Now if z is a periodic point of f_c, then necessarily the orbit of z is bounded and therefore by property 4.2, $|z| \le |c| + 1$. Again $J(f_c)$ is the smallest closed set containing all repelling periodic points of f_c, therefore, any $z \in J(f_c)$ also has the property that $|z| \le |c| + 1$. Now by property 4.3, the iterates of $J(f_c)$ are precisely $J(f_c)$. Hence the iterates of z are also bounded by $|c| + 1$. ■

Proposition 4.5: The Julia set J_c of the function f_c is compact for all $c \in \hat{C}$.

Proof: By theorem 1.1, there exist a real number r such that for $|z| \ge r$,

$$|f_c(z)| \ge 2z$$

Thus, for $|z| \ge r$, $|f_c^n(z)| \to \infty$ as $n \to \infty$ Since, Julia set J_c of f_c is the boundary of the set of those points of $\hat{C}$ which do not converge to ∞ under iteration of f_c, so we must have

$J_c \subseteq S_r(0)$ I.e. the sphere of radius r and centre at origin

$\Rightarrow |z| \le r$ For all $z \in J_c$

Which shows that J_c is bounded.

Next we need to show that J_c is closed.

Let $z \in B(\infty)$, the basin of attraction of ∞ under f_c. Therefore, there exist $k \in N$ such that $f_c^k(z) \ge r$. Thus, to each $z \in B(\infty)$, there exist an open sphere $S_\varepsilon(z)$ such that

$$z \in S_\varepsilon(z) \;\Rightarrow\; f^k(z) \to \infty \quad \text{As } k \to \infty$$

$$\Rightarrow z \in B(\infty)$$

Therefore, $z \in S_\varepsilon(z) \subseteq B(\infty)$ which shows that $B(\infty)$ is an open set.

Now as $B(\infty) = \hat{C} - K(f_c)$, so the filled in Julia set $K(f_c)$ is closed. Again the boundary of $K(f_c)$ is the Julia set J_c we have that J_c is closed.

Hence J_c is compact. ■

Proposition 4.6: J_c is symmetric about the origin for all $c \in \mathbb{C}$.

Proof: Suppose $z_o \in J_c$. By invariant property of Julia set it is clear that

$$f_c(z_0) \in J_c \Rightarrow z_0^2 + c \in J_c \Rightarrow f_c^{-1}(z_0^2 + c) \in J_c \Rightarrow \{z_0, -z_0\} \in J_c \Rightarrow -z_0 \in J_c$$

Hence the Julia set J_c of f_c is symmetric about the origin. ■

Proposition 4.7: For any real values of c, the Julia set J_c is symmetric about both real and imaginary axes.

Proof: Recall that any set $S \subset C$ is symmetric about the real axis if the conjugate of each element of S belongs to the set S i.e. $z \in S \Rightarrow \bar{z} \in S$. We first show that $f_c^n(\bar{z}) = \overline{f_c^n(z)}$.

Let $z_1 = a_1 + ib_1$

Now, $f_c(z_1) = a_1^2 - b_1^2 + c + 2a_1 b_1 i = a_2 + ib_2$ (say)

$$f_c(\bar{z}) = a_1^2 - b_1^2 + c - 2a_1 b_1 i = a_2 - ib_2 = \overline{f_c(z_1)}$$

Suppose $f_c^{r-1}(\bar{z}_1) = \overline{f_c^{r-1}(z_1)}$ and $f_c^{r-1}(z_1) = a_r + ib_r$

Now, $f_c^r(\bar{z}_1) = f_c\left(f_c^{r-1}(\bar{z}_1)\right) = f_c\left(\overline{f_c^{r-1}(z_1)}\right) = a_r^2 - b_r^2 + c - 2a_r b_r i = \overline{f_c^r(z_1)}$

Thus, by induction,

$$f_c^n\left(\bar{z}\right) = \overline{f_c^n(z)} \text{ For all } n \in N$$

Now suppose,

$z \in K_c$, the filled in Julia set of f_c

$\Rightarrow \{f_c^n(z)\}_{n=1}^{\infty}$ Is bounded

$\Rightarrow \left\{\overline{f_c^n(z)}\right\}_{n=1}^{\infty}$ Is bounded

$\Rightarrow \left\{f_c^n(\bar{z})\right\}_{n=1}^{\infty}$ Is bounded $\Rightarrow \bar{z} \in K_c$

Thus K_c and hence its boundary J_c is also symmetric about the real axis.

Symmetry of J_c about the imaginary axis immediately follows from the fact that it is symmetric about the real axis as well as the origin. ■

5. VISUALIZING JULIA SETS

To see how one can plot Julia sets, one will look at the simplest method called "The escape criterion method", which generates the filled in Julia sets. Julia sets do not have interiors; however, this method often ends up plotting the interior along with the Julia set. This method is based on the following facts:

At each step of iteration of the map $f_c(z) = z^2 + c$, geometrically we square the modulus and double the angle of the number to which it operates and then shift by the parameter c. If c is small the squaring part dominates the shift so the behavior is not very different from the case $c = 0$, discussed above. Just as this case, two types of behavior are possible depending on the starting point (seed). The first type of points lie in the basin of attraction of the fixed points or the attracting cycles and the orbits of the second type tend to infinity. The Julia set separates these two very different types of behavior. If a point is in the Julia set, there are arbitrarily close points that iterates to infinity but also arbitrarily close points that do not wander far off under iteration. So, our first method for plotting the Julia set is to color points with a color if their orbits tend to infinity and to color them by another one (contrasting to the first color) if they do not. The boundary between these two regions is then the Julia set.

Now, our problem is converted to find out those points whose orbits tend to infinity which is tackled by the following theorem:

Theorem 5.1: If $|c| \leq 2$, Then the orbit of the points lie outside the circle of radius 2 i.e. the set of points $\{z: |z| > 2\}$, escape to infinity.

Proof: Let $E = \{z : |z| > 2\}$. Then for any $z \in E$ we have,

$$|z| > |c|$$

Now, $|z^2| = |c + z^2 - c| \leq |c| + |z^2 - c|$

$$\Rightarrow |z^2| - |c| \leq |z^2 - c| \Rightarrow |z^2| - |z| < |z^2 - c|$$

Replacing c by $-c$,

$$(|z| - 1)|z| < |z^2 + c| = |f_c(z)|$$

Since $|z| > 2$, therefore, $|z| - 1 > 1$ and hence one can write $|z| - 1 = 1 + \delta$ where $\delta > 0$.

$$\therefore \quad |z| < (1 + \delta)|z| < |f_c(z)|$$

Thus, z will move further from origin by the action of f_c

Again, $\left| f_c^{2}(z) \right| = \left| f_c\left(f_c(z) \right) \right| > (1+\delta)\left| f_c(z) \right| > (1+\delta)^2 \left| z \right|$

Therefore, $f_c^{2}(z)$ is even further from the origin than $f_c(z)$.

Continuing in this fashion, one can find

$$\left| f_c^{n}(z) \right| > (1+\delta)^n \left| z \right|$$

Recall that $\delta > 0$, so that $1 + \delta > 1$. Hence the number $(1+\delta)^n \to \infty$ as $n \to \infty$ so it follows that $\left| f_c^{n}(z) \right| \to \infty$ as $n \to \infty$ i.e. the orbit of z tend to infinity. ■

In view of the above theorem the algorithms that produce the Julia sets shown in the figure 5.1 and 5.2 is as follows:

Step 1:

Input c_1 and c_2 where $c = c_1 + i c_2$

Step 2:

Select a 200×200 grid in the plane.

Step 3:

For each point z_0 in this grid, compute the first 20 points on the orbit of z_0. Check at each point of the iteration whether the corresponding iterated value lies outside the circle of radius 2.

Step 4:

If any point on the orbit lies outside the circle of radius 2, then stop iteration and color the point z_0 by purple color.

Step 5:

If all 20 points on the orbit lie outside the circle of radius 2, then keep the original point z_0 without any color (i.e. keep it white)

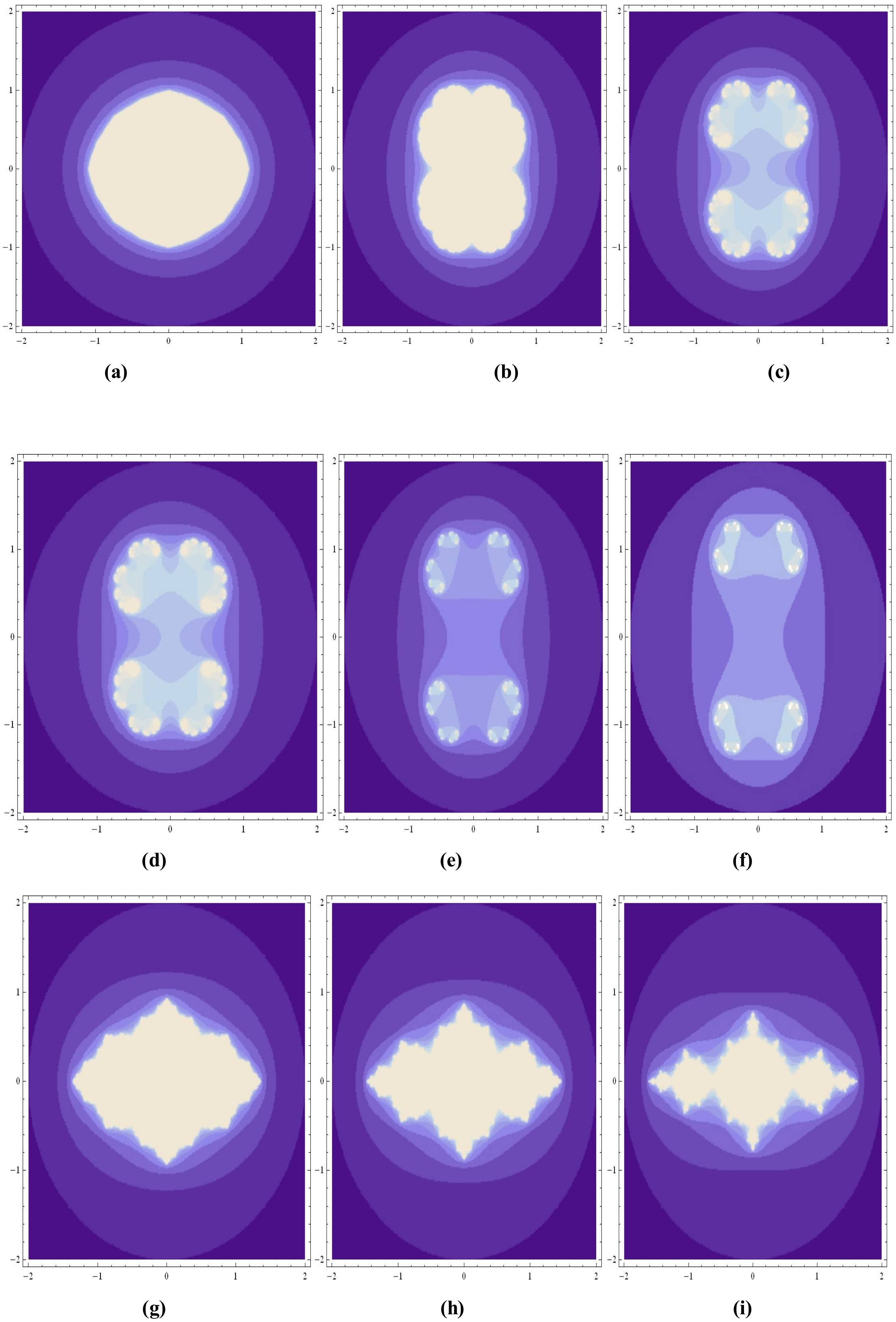


Figure 5.1: Filled in Julia Set for the Parameter (*c*) Values

$(a) - 0.1,\ (b)\, 0.25,\ (c)\, 0.4,\ (d)\, 0.65,\ (e)\, 0.8,\ (f)\, 0.9,\ (g) - 0.5,\ (h) - 0.7,\ (i) - 1$)

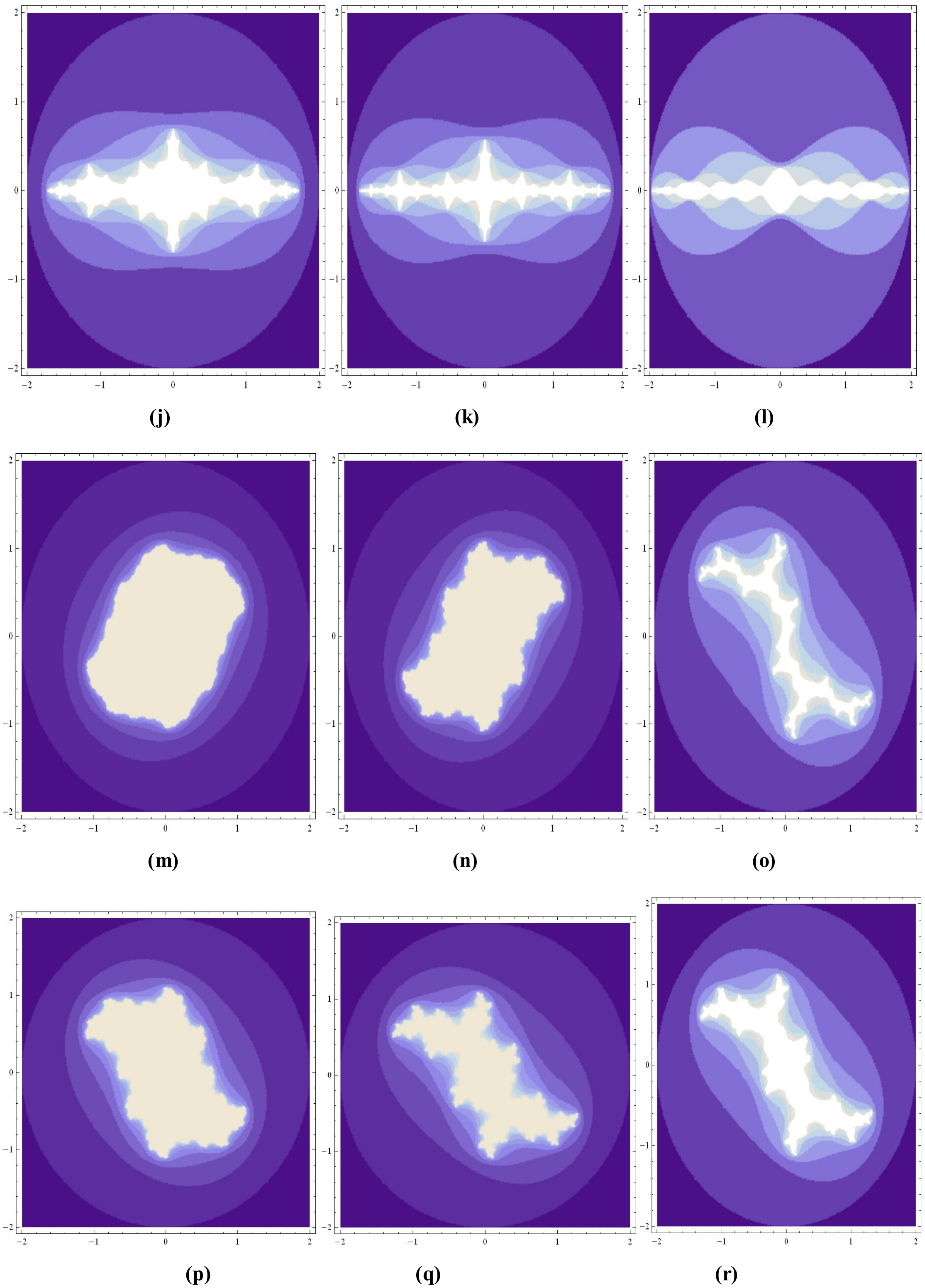


Figure 5.2: Filled in Julia Set for the Parameter (*c*) Values

$(j) -1.25,\ (k)\ -1.5,\ (l)\ -1.9, (m)\ -0.3i,\ (n)\ -0.5i,\ (o)\ i,\ (p)\ 0.1+0.5i, (q)\ -0.1+0.75i,\ (r) -0.10.9i$).

Though this method gives beautiful pictures of Julia sets, it actually gives the interior together with the Julia sets i.e. filled in Julia sets instead of showing the actual Julia sets alone. Another disadvantage of the algorithm is that it takes more computational time. To overcome these disadvantages, there are various computation methods for visualizing Julia sets in computer screen such as Boundary Scanning Method (BSM), Inverse Iteration Method (IIM), and Modified Inverse Iteration Method (MIIM) etc. These methods are explained and compared in [8], [19] and [3]. Among these MIIM is the best method for both computation time and remarkable picture quality. This method is discussed in detailed by Peitgen and Richter [15]. The algorithm for this method is as follows:

Step 1:

Choose any real number c such that $-2 \le c \le 2$.

Step 2:

Find any repelling point x_0 of f_c as the starting point and set $S_0 = \{x_0\}$.

Step 3:

Choose the number of iteration, say n .

Step 4:

Find the image of S_0 under g_1 and g_2 where $g_1(z) = \sqrt{z-c}$, and $g_2(z) = -\sqrt{z-c}$

Step 5:

Set $S_1 = g_1[S_0] \cup g_2[S_0]$.

Step 6:

Find the image of $S_1 - S_0$ under g_1 and g_2 and set $S_2 = g_1[S_1 - S_0] \cup g_2[S_1 - S_0]$.

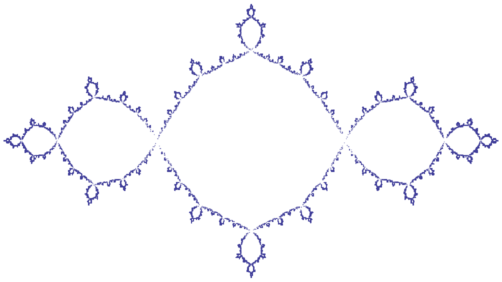
Step 7:

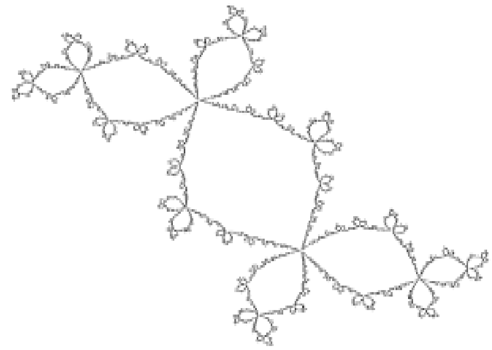
Repeat the process for n times and finally get $S_n = g_1[S_{n-1} - S_{n-2}] \cup g_2[S_{n-1} - S_{n-2}]$

Step 8:

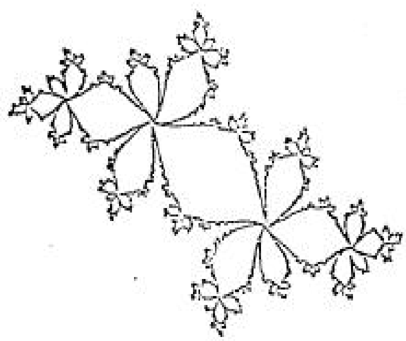
The set $\bigcup_{i=0}^{n} S_i$ will give the Julia set for f_c approximately.

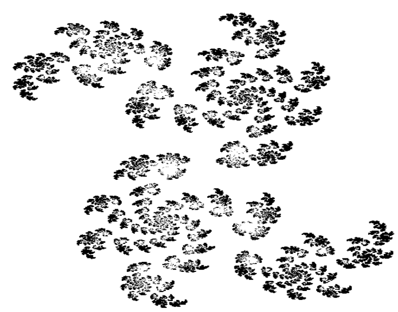
Using this algorithm Julia sets for the parameter value $c=-1$, $c=0.123+0.745i$, $c=-0.5+0.55i$, $c=0.4+0.3i$ And $c=i$ are drawn which are shown in the figure 5.3 below:



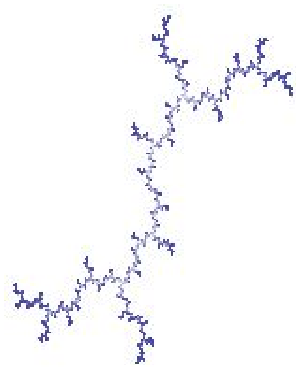


(a) (b)





(c) (d)



(e)

Figure 5.3: Julia Sets for the Parameter (*c*) Values

(a) -1, (b) $0.123+0.745i$, (c) $-0.5+0.55i$, (d) $0.4+0.3i$ (e) i

6. CONCLUSIONS

For all parameter values c such that $|c| < 0.25$, effect of shift by c does not change the picture of the respective Julia set much. In this case the Julia set is a quasi circle or a loop. As soon as the magnitude of c increases the shifting part of the function become significant, its picture becomes strange. For example, the Julia set for the parameter value $c = -1$ consists of a central loop surrounded by many smaller touching loops which in turn are surrounded by even smaller loops, and so on, see figure 5.3(a). Moreover, the Julia set for the parameter value c close to -1 has a similar structure of touching loops i.e. the configuration is stable for $c = -1$. The sets for other parameter values yields more surprises. For c close to $-0.123 + 0.745i$ the Julia set is again formed by a hierarchy of loops, but now they meet three at a point as shown in figure 5.3(b). For this case f_c has an attracting period-3 orbit. Further, increasing the value of c nearer to $-0.5 + 0.55i$, the loops meet five at a point, figure 5.3(c). Again, choosing the value of c as $-i$, the loops collapse and the Julia set takes a dendrite or 'twig-like' shape, see figure 5.3(e). As the value of c becomes larger a dramatic change in the nature of the Julia set takes place. For example, when $c = 0.4 + 0.3i$ the connectedness of the Julia set completely breaks up and it become a totally disconnected one, figure 8(d). In this situation, the Julia set becomes a fractal dust and is identical to the filled in Julia set. Any point not in the Julia set of this type iterates to infinity, as no other destination is possible.

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